International Journal of Diophantine Approximation and Number Theory
Vol. 1 No. 1 (January-June, 2026)

Continued β-fractions in the field of formal power series

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Abstract

A formal power series f can be represented using the β-transformation $T'_\beta : f \mapsto \frac{1}{f} - [\frac{1}{f}]_\beta$ where $|\beta| > 1$. So, f is expanded into $[A_0, A_1, A_2, \ldots]_\beta$ where $(A_i)_{i \geq 0}$ are β-polynomials. This later expression is called continued β-fraction expansion of f. The aim of this paper is to characterize the formal power series with a finite continued β-fraction where β is a unit Pisot series.

Classification Mathematic Subject: 11R06, 37B50.
Key words: Formal power series, β-expansion, unit Pisot , continued β-fraction.

1 Introduction

The continued fractions has been appeared in 1655 and more deeply studied by Euler how was populated this theory. This representation of a real number x has this form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} := [a_0, a_1, \ldots, a_n \ldots]$$

where a_i are integers for all $i \geq 0$ and $a_i > 0$ for $i \geq 1$ defined by $a_i = \left[\frac{1}{T^{i-1}(x)}\right]$ such that T is the transformation defined by $T(x) = \frac{1}{x} - [\frac{1}{x}]$. The a_i are called the partial quotients. This continued fraction is finite if and only if $x \in \mathbb{Q}$.

Moreover, an other expansion theory, the β-expansion, of a real number x has been introduced by A. Rényi [9]. Since then, its arithmetic, diophantine and ergodic properties

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has been extensively studied by several authors. For a real number $\beta > 1$, the β-expansion of a real number $x < 1$ is defined as follows:

$$d_\beta(x) = x_1 x_2 x_3 ...$$

where $x_i = [\beta T_\beta^{i-1}(x)]$ such that T_β is the β-transformation defined by $T_\beta(x) = \beta x - [\beta x]$. Thus, we have $x = \sum\limits_{i\geq 1} \frac{x_i}{\beta^i}$.

For $x > 1$, since $\beta > 1$, there exists an unique integer $n > 0$ such that $\beta^{n-1} \leq x < \beta^n$. So, we can write $\frac{x}{\beta^n} = \sum\limits_{i\geq 1} \frac{y_i}{\beta^i}$ where $(y_i)_{i\geq 1}$ is the β-expansion of $\frac{x}{\beta^n}$. Thus, we have

$$x = \sum_{i=-n}^{\infty} x_i \beta^{-i} \quad \text{with} \quad x_i = y_{i-n}.$$

We define the β-integer part of x by $[x]_\beta = \sum\limits_{i=-n}^{0} x_i \beta^{-i}$ and the β-fractional part of x by $\{x\}_\beta = \sum\limits_{i>0} x_i \beta^{-i} = x - [x]_\beta$. If $\{x\}_\beta = 0$, then x is called β-integer and the set of β-integers is denoted by $\mathbb{Z}_\beta$.

Enomoto has introduced a similar algorithm of continued fractions by changing the decimal base to obtain a new expansion of real numbers that is the continued fraction in base ϕ where ϕ is the golden number: $\phi = \frac{1+\sqrt{5}}{2}$. This expansion is a generalization of Euclide 's algorithm where the sequence of partial quotients consists of ϕ-integer. From this algorithm J. Bernat [3] has proved that continued ϕ-fraction of x is finite if and only if $x \in \mathbb{Q}(\phi)$ (the smallest fields containing $\mathbb{Q}$ and ϕ).But given a complete characterization of all bases seems to be very hard to achieve in the real case.

In this paper, we consider an analogue of this concept in algebraic function over finite fields. The aim of this paper is to prove that any element of $\mathbb{F}_q(x, \beta)$ has a finite continued β-fraction where β is a Pisot unit series.

2 Field of formal series $\mathbb{F}_q((x^{-1}))$

Consider the finite field $\mathbb{F}_q$ where p is a prime number. We denote by $\mathbb{F}_q[x]$ the ring of polynomials with coefficients in $\mathbb{F}_q$ and $\mathbb{F}_q(x)$ the field of fractions of $\mathbb{F}_q[x]$. Let $\mathbb{F}_q((x^{-1}))$ be the field of formal power series:

$$\mathbb{F}_q((x^{-1})) = \left\{ f = \textstyle\sum_{i\geq n_0} f_i x^{-i} \ : \ n_0 \in \mathbb{Z} \quad \text{and} \quad f_i \in \mathbb{F}_q \right\}.$$

Let us define the degree of $f = \sum f_i x^{-i}$ by $\deg(f) = \sup\{-i : f_i \neq 0\}$ if $f \neq 0$ and $\deg(0) = -\infty$. Thus, we define $|f| = q^{\deg(f)}$. Note that $|.|$ is a not archimedean absolute value over $\mathbb{F}_q((x^{-1}))$. We also define the polynomial part of f by $[f] := \sum_{i\leq 0} f_i x^{-i}$.

An element $\beta \in \mathbb{F}_q((x^{-1}))$ is called algebraic integer over $\mathbb{F}_q[x]$ with degree d if its minimal polynomial has the following form:

$$P(y) = y^n + A_{n-1} y^{n-1} + \cdots + A_0 \text{ where } A_i \in \mathbb{F}_q[x].$$

When $A_0 \in \mathbb{F}_q^*$, β is called unit.

An element $\beta = \beta_1$ is called Pisot series if it is an algebraic integer over $\mathbb{F}_q[x]$, $|\beta| > 1$ and $|\beta_j| < 1$ for all Galois conjugates β_j. Since $\mathbb{F}_q[x] \subset \mathbb{F}_q((x^{-1}))$, every algebraic element can be valuated. However, since $\mathbb{F}_q((x^{-1}))$ is not algebraically closed, the Galois conjugates of β need not necessarily be a power series.

The following theorem, proved in [1], characterize the Pisot power series:

Theorem 2.1. *Let $\beta \in \mathbb{F}_q((x^{-1}))$ be an algebraic integer over $\mathbb{F}_q[x]$ and its minimal polynomial be*

$$P(y) = y^n + A_{n-1}y^{n-1} + \cdots + A_0, \quad A_i \in \mathbb{F}_q[x]$$

Then β is a Pisot element if and only if $|A_{n-1}| > \max_{i \neq n-1} |A_i|$.

2.1 continued fraction expansion in $\mathbb{F}_q((x^{-1}))$

As in the classical context of real numbers, we have a continued fraction algorithm in $\mathbb{F}_q((X^{-1}))$. Let T the transformation defined by:

$$\begin{aligned} T : M_q &\to M_q \\ f &\mapsto \tfrac{1}{f} - [\tfrac{1}{f}]. \end{aligned}$$

Then, for any $f \in M_q$, we have:

$$f = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = [0, a_1, a_2, \ldots]$$

where $(a_i)_{i \geq 0} \in \mathbb{F}_q[X]$ defined, for any positive integer n, by: $a_n = \left[\dfrac{1}{T^{n-1}(f)}\right]$.

Let $f \in \mathbb{F}_q((X^{-1}))$ and $a_0 = [f]$, we have:

$$f = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = [a_0, a_1, a_2, \ldots].$$

The later expression is called continued fraction expansion of f and the sequence $(a_i)_{i \geq 0}$ is called the sequence of partial quotients of f.
Define two sequences of polynomial $(p_n)_{n \in \mathbb{N}}$ and $(q_n)_{n \in \mathbb{N}}$ by:

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$$

So $p_0 = a_0$, $q_0 = 1$, $p_1 = a_0 a_1 + 1$, $q_1 = a_1$ and for any $n \geq 2$,

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

We easily check that

$$P_nQ_{n-1} - P_{n-1}Q_n = (-1)^{n-1}$$

and

$$\frac{P_n}{Q_n} = [a_0, a_1, a_2, \ldots, a_n].$$

The rational fraction $\frac{P_n}{Q_n}$ is called the $n^{th}-$convergent of f. We have the following important equality

$$\left|f - \frac{P_n}{Q_n}\right| = |a_{n+1}|^{-1}|Q_n|^{-2}.$$

If $\frac{P_n}{Q_n}$ is a convergent of f, then we have $\left|f - \frac{P_n}{Q_n}\right| \leq \frac{1}{|Q_n|^2}$. So, the best rational approximation of f are its convergents

2.2 β-expansion in $\mathbb{F}_q((x^{-1}))$

Let β, $f \in \mathbb{F}_q((x^{-1}))$ with $|\beta| > 1$. A representation in base β (or β-representation) of a formal series $f \in M_q = \{f \in \mathbb{F}_q((X^{-1})) : |f| < 1\}$ is an infinite sequence $(x_i)_{i\geq 1}$, $x_i \in \mathbb{F}_q[x]$, such that

$$f \ = \ \sum_{i\geq 1} x_i\beta^{-i}.$$

A particular β-representation of f, noted $d_\beta(f)$, is called the β-expansion of f in base β and obtained by using the β-transformation T_β defined by:

$$\begin{aligned} T_\beta : M_q &\to M_q \\ f &\mapsto \beta f - [\beta f] \end{aligned}$$

Then, for any $f \in M_q$, we have $d_\beta(f) = 0.x_1x_2x_3...$ where $(x_i)_{i\geq 1}$ are polynomials defined, for any positive integer n, by

$$x_i = \left[\beta T_\beta^{i-1}(f)\right].$$

For better characterization of β-expansion, M.Hbaib and M.Mkaouar in [6] showed the following theorem.

Theorem 2.2. *Let $(a_i)_{i\geq 1}$ be a β-representation of $f \in M_q$. Then, $(a_i)_{i\geq 1}$ is the β-expansion of f if and only if $|a_i| < |\beta|$ for $i \geq 1$.*

Now let . By a similar way of the case of real numbers, we deduce $d_\beta(f)$ for all $f \in \mathbb{F}_q((x^{-1}))$ with $|f| \geq 1$. When there are only finitely many non-zero elements in $d_\beta(f)$, we say that f has a finite β-expansion. In this case, we omit the ending of consecutive zeros and the set of power series having a finite β-expansion is denoted by $\mathrm{Fin}(\beta)$. In [5], K. Scheicher has identified this set as $\mathbb{F}_q[x, \beta^{-1}]$ (the minimal ring containing x and β^{-1}) when β is a Pisot series.

Remark 2.3. *If β is a unit Pisot series, so $\mathbb{F}_q[x, \beta^{-1}] = \mathbb{F}_q[x, \beta] = \mathrm{Fin}(\beta)$.*

We define the β-polynomials part of f by $[f]_\beta = \sum_{i=-n}^{0} x_i \beta^{-i}$ and the β-fractional part of f by $\{f\}_\beta = \sum_{i>0} x_i \beta^{-i} = f - [f]_\beta$. If $\{f\}_\beta = 0$, then f is called β-polynomial and the set of β-polynomials is denoted by $(\mathbb{F}_q[x])_\beta$. We define also the set of β-fractions denoted by $(\mathbb{F}_q(x))_\beta$ of power series which can be taken as a fraction of two β-polynomials. We can easy see that: $(\mathbb{F}_q[x])_\beta \subseteq \mathbb{F}_q[x, \beta]$ and $(\mathbb{F}_q(x))_\beta \subseteq \mathbb{F}_q(x, \beta)$.

3 continued β-fraction expansion algorithm

In this section we explain how to generalize the algorithm of the expansion in continued fraction in the field of formal power series in base $\beta \in \mathbb{F}_q((x^{-1}))$ with $|\beta| > 1$. This study is very similar to the classical one with usual continued fractions.
Let T'_β be the β-transformation defined by:

$$\begin{aligned} T'_\beta : M_q &\to M_q \\ f &\mapsto \tfrac{1}{f} - [\tfrac{1}{f}]_\beta. \end{aligned}$$

Then, for any $f \in M_q$, we have

$$f = \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{\ddots}}} = [0, A_1, A_2, \ldots]_\beta,$$

where $(A_i)_{i\geq 1}$ are β-polynomials defined, for any positive integer i, by:

$$A_i = \left[\frac{1}{T'^{(i-1)}_\beta (f)} \right]_\beta$$

Let $f \in \mathbb{F}_q((X^{-1}))$ and $A_0 = [f]_\beta$, we have

$$f = A_0 + \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{\ddots}}} = [A_0, A_1, A_2, \ldots]_\beta,$$

The later expression is called continued β-fraction expansion of f, the sequence $(A_i)_{i\geq 0}$ is called the sequence of partial β-quotients of f and the quantities f_n defined by:
$f = [A_0, A_1, A_2, \ldots, f_n]_\beta$ are called n^{th} complete β-quotients.
We define two sequences $(P_n)_{\ n\in\mathbb{N}}$ and $(Q_n)_{\ n\in\mathbb{N}}$ in $\mathbb{F}_q[x, \beta]$ by:

$$\begin{cases} P_0 = A_0, Q_0 = 1 \\ P_1 = A_0 A_1 + 1, Q_1 = A_1 \\ P_n = A_n P_{n-1} + P_{n-2}, Q_n = A_n Q_{n-1} + Q_{n-2} \forall n \geq 2 \end{cases}$$

$\dfrac{P_n}{Q_n} = [A_0, A_1, A_2, ..., A_n]_\beta$ is called the n^{th} β-reduced of f.

A similar way of the usual continued fractions, allow us to prove this theorem:

Theorem 3.1. *The sequence of β-reduced of $f \in \mathbb{F}_q((x^{-1}))$ is converging to f.*

Proof. Let $f = [A_0, A_1, ..., A_n, A_{n+1}, ...]_\beta$ and let g be its n^{th} complete β-quotients, so

$$\begin{aligned} f &= [A_0, A_1, ..., A_n, g]_\beta = \frac{gP_n + P_{n-1}}{gQ_n + Q_{n-1}} \\ Q_n f &= P_n(\frac{gQ_n + Q_{n-1}}{gQ_n + Q_{n-1}}) + \frac{P_{n-1}Q_n - P_nQ_{n-1}}{gQ_n + Q_{n-1}} \\ &= P_n + \frac{(-1)^n}{gQ_n + Q_{n-1}} \end{aligned}$$

Since $|g| > |A_{n+1}| > |\beta|$ and $|Q_n| < |Q_{n+1}|$ we obtain

$$|f - \frac{P_n}{Q_n}| < \frac{1}{|\beta||Q_n|^2}$$

Let $f \in \mathbb{F}_q((x^{-1}))$, we call that f is locally β-approximation by elements of $\mathbb{F}_q(x, \beta)$ when there exist an unfinite elements $\frac{P}{Q}$ of $\mathbb{F}_q(x, \beta)$, $A \in \mathbb{F}_q^*$, an integer $r > 0$ such that

$$|Qf - P| < \frac{A}{|\beta||Q|^r}$$

In fact, the β-reduced are characterized by the locally best β-approximation property given in this proposition:

Proposition 3.1. *Let $f \in \mathbb{F}_q((x^{-1}))$, $\frac{P_n}{Q_n}$ be its n^{th} β-reduced and $S \in \mathbb{F}_q[x, \beta]$ be such that $|S| < \frac{|Q_n|}{|\beta|}$ then, for all $R \in \mathbb{F}_q[x, \beta]$, $|Q_nf - P_n| < |Sf - R|$.*

Proof. To see this, suppose without loss of generality that $|Q_{n-1}| < |S| < |Q_n|$ and note that, because $P_nQ_{n-1} - P_{n-1}Q_n = (-1)^{n-1}$, the matrix $\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix}$ is unimodular, so there exist two β-polynomial A and B such that

$$S = AQ_n + BQ_{n-1} \text{ and } R = AP_n + BP_{n-1}.$$

Then,

$$Sf - R = A(Q_nf - P_n) + B(Q_{n-1}f - P_{n-1})$$

and the evident fact that $|B| > |A| > |\beta| > 1$ shows that, indeed

$$|Sf - R| > |Q_{n-1}f - P_{n-1}| > |Q_nf - P_n|$$

Proposition 3.2. *Let P and $Q \in \mathbb{F}_q[x,\beta]$ such that $|Qf - P| < \frac{1}{|\beta||Q|}$. Then, $\frac{P}{Q}$ is a β-reduced of f.*

Proof. Let $A \in \mathbb{F}_q[x,\beta]$ such that $|A| < |Q|$. There exist $B \in \mathbb{F}_q[x,\beta]$ such that

$$\frac{1}{|\beta|} < |QB - PA| = |A(Qf - P) - Q(Af - B)|$$

Since $|A(Qf-P)| < \frac{1}{|\beta|}$, we have $|Q(Af-B)| > \frac{1}{|\beta|}$. Whence, $|Af-B| > \frac{1}{|\beta||Q|} > |Qf-P|$ this entails that $\frac{P}{Q}$ is indeed a locally best β-approximation and by Proposition 3.1, $\frac{P}{Q}$ is a β-reduced of f.

4 Finiteness of continued β-fraction expansion

The aim of this section is to prove the following theorem:

Theorem 4.1. *Let $f \in \mathbb{F}_q((x^{-1}))$ and β be a Pisot unit power formal series, then f has a finite continued β-fraction if and only if $f \in \mathbb{F}_q(x,\beta)$.*

It is clear that a finite continued β-fraction represents an element of $\mathbb{F}_q(x,\beta)$, and it is natural to ask whether the reciprocal property holds.

We remark that we need first to define a canonical way to expand elements of $\mathbb{F}_q(x,\beta)$. The following proposition allows us to expand any element of $\mathbb{F}_q(x,\beta)$ as a β-fraction.

Proposition 4.1.

Let β be a unit Pisot series, then $\mathbb{F}_q(x,\beta) = (\mathbb{F}_q(x))_\beta$

Proof.
Let $f \in \mathbb{F}_q(x,\beta)$, so there exist A_i ab=nd B_i both belong to $\mathbb{F}_[x]$ such that

$$f = \frac{\sum_{i=0}^{n} A_i\beta^i}{\sum_{i=0}^{m} B_i\beta^i}$$

where $n, m < deg(\beta)$.
Since β is a unit Pisot series and by Remark 2.3, we have $\sum_{i=0}^{n} A_i\beta^i$ and $\sum_{i=0}^{m} B_i\beta^i$ in $Fin(\beta)$, so

$$f = \frac{\sum_{i=-r}^{p} P_i\beta^i}{\sum_{i=-s}^{q} Q_i\beta^i}$$

where $p, q < deg(\beta)$ and $|P_i| < |\beta|$ and $|Q_i| < |\beta|$.
Let $k = max(r, s)$ so

$$f = \frac{\sum\limits_{i=-r}^{p} P_i \beta^{i+k}}{\sum\limits_{i=-s}^{q} Q_i \beta^{i+k}}$$

hence $f \in (\mathbb{F}_q(x))_\beta$.

Proof of Theorem 4.1. Let $f \in \mathbb{F}_q(x, \beta) = (\mathbb{F}_q(x))_\beta$ (by Proposition 4.1), so there exist A and B both belong to $(\mathbb{F}_q[x])_\beta$ such that $f = \frac{A}{B}$.
We have $[Bf]_\beta = [A]_\beta = Bf = A$ hence $\{Bf\}_\beta = \{A\}_\beta = 0$.
This implies that

$$|Bf - [Bf]_\beta| = 0 < \frac{1}{|\beta||B|}$$

By Proposition 3.2, we deduce that $\frac{A}{B}$ is a β-reduced of f.
Then, $f = \frac{P_n}{Q_n} = [A_0, ..., A_n]$ which is a finite continued β-fraction expansion.

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