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Multidimensional Continued Fractions over the Field of Formal Power Series

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Abstract

We study multidimensional continued fraction algorithms over the field of formal power series. In this case, we prove that the simultaneous diophantine properties of the algorithms is stronger than in the real case. Jacobi-Perron algorithm is exponentially strong convergent, on the other hand Brun algorithm is only strong convergent. Therefore their convergence is better than in the real case. We also establish a relation between the Jacobi-Perron algorithm and the version of it introduced by Dubois. Regarding the periodicity of the Jacobi-Perron algorithm, we define periodic vectors whose coordinates belong to certain finite degree extension fields.

Keywords: Continued fractions, Algorithm.

1 Introduction

Multidimensional continued fractions have been extensively studied over the real field and the reader can find many references in the subject on the book by Schweiger [S2]. A version of the so called Jacobi-Perron algorithm has been studied also over the field of formal power series, and it was introduced by Dubois [D1], as far as we know. The aim of this paper is to pursue the study of multidimensional continued fractions over the field of formal power series.

For completeness we recall that was Jacobi [J] who introduced the notion of multidimensional continued fractions. His aim was to obtain a generalization of the Lagrange Theorem (see Hardy-Wright [HW], p. 144) in the cubic case.

Lagrange Theorem. Let x be a real number and $x = [a_0; a_1, \ldots]$ be its continued fraction expansion. The continued fraction expansion of an irrational x is ulteriorly periodic if, and only if, x is a quadratic surd.

Subsequently the Jacobi algorithm was studied by Perron [P] who besides generalizing its definition to the multidimensional case, proved a central property: its convergence in every dimension (see (4.1)). That is the reason why nowdays the map is called Jacobi-Perron algorithm. For further references on the subjet see also Bernstein [B] and Schweiger [S1]. It is still an open question whether the Jacobi-Perron algorithm is periodic when applied to a n-tuple $(f_1, \ldots, f_n)$, where $f_1, \ldots, f_n$ belong to a real number field of degree $n+1$. This is also conjectured in the case of the Jacobi-Perron algorithm defined over finite fields for a field of formal power series. Regarding this case, in the end of *Section* 5, we propose a weaker conjecture which concerns Pisot elements.

The article is organized in the following way. In *Section* 2 we present a brief review of the fields of formal power series. There we present the basic notions need throughout the work. In *Section* 3 the definition of a general multidimensional continued fraction algorithm over a field of formal power series is given, and we find the notions of convergence that we need (see $(3.1-2)$). The Jacobi-Perron algorithm J is studied in *Section* 4. It associates to every n-tuple $f \in \mathbb{F}_q((X^{-1}))^n$ a sequence of $(n+1)$-tuple of polynomials $(A_0^{(t)}, A_1^{(t)}, \ldots, A_n^{(t)}) \in (\mathbb{F}_q[X])^{n+1}$. We prove that J is exponentially strong convergent.

Theorem 1.1 *Let* $(f_1, \ldots, f_n) \in M^\star$*, then, for all* $1 \leq i \leq n$*,*

$$A_0^{(t)} f_i - A_i^{(t)} \text{ converges exponentially to } 0, \text{ as } t \to +\infty.$$

The version of Jacobi-Perron algorithm D introduced in [D1] is presented in *Section* 5. We show a relation between the algorithms J and D.

2 The field of formal power series

For completenes, we include some basic notions of the field of formal power series. Let $\mathbb{F}_q$ be a field with q elements of characteristic p, $\mathbb{F}_q[X]$ the set of polynomials of coefficients in $\mathbb{F}_q$ and $\mathbb{F}_q(X)$ its field of fractions. The set $\mathbb{F}_q((X^{-1}))$ is the field of formal power series over $\mathbb{F}_q$

$$\mathbb{F}_q((X^{-1})) = \{\omega = \sum_{j=s}^{+\infty} a_j X^{-j} \ : \ a_j \in \mathbb{F}_q, \ s \in \mathbb{Z}\}.$$

Let $\omega = \sum_{j=s}^{+\infty} a_j X^{-j} \in \mathbb{F}_q((X^{-1}))$, where $a_s \neq 0$. We denote its polynomial part by $[\omega]$ and $\{\omega\}$ its fractional part. We remark that $\omega = [\omega] + \{\omega\}$. We define a non archimedean absolute value on $\mathbb{F}_q((X^{-1}))$ by $| \omega |= e^{-s}$. It is clear that, for all $P \in \mathbb{F}_q[X]$, $| P |= e^{\deg P}$ and, for all $Q \in \mathbb{F}_q[X]$, such that $Q \neq 0$, $| \frac{P}{Q} |= e^{\deg P - \deg Q}$.

Let $E = (\mathbb{F}_q((X^{-1})))^n$, E is a vectorial space over $\mathbb{F}_q((X^{-1}))$. We define a norm over E as follows, for all $f = (f_1, \ldots, f_n) \in E$,

$$\| f \|= \max_{1\leq i\leq n} | f_i |.$$

Let $A_1, \ldots, A_m \in E$, then we can verify that

$$\| A_1 + \cdots + A_m \|\leq \max_{1\leq i\leq m} \| A_i \|.$$

3 Multidimensional continued fractions

Let $B \subset E$ and $T : B \longrightarrow B$ be a map. The pair (B, T) is called a fibred system if there exists a finite or countable partition $\{B(P) : P \in I\}$ of B, where $I \subset \mathbb{F}_q[X]^n$, such that the restriction of T to any $B(P)$ is an injective map. As E is a normed space, we assume that a system defines an algorithm of multidimensional continued fractions if for all $P = (P_1, \ldots, P_n) \in I$, there exists an $(n+1)\times(n+1)$ invertible matrix $\alpha(P) = (C_{i,j})$ with entries in $\mathbb{F}_q[X]$ such that, if $y = T(f)$, where $f \in B(P)$, then $y_i = \dfrac{C_{i0} + \sum_{j=1}^n C_{ij} f_j}{C_{00} + \sum_{j=1}^n C_{0j} f_j}$, for all $1 \leq i \leq n$.

The map T is called a *multidimensional continued fraction algorithm*. For all $1 \leq i \leq n$, if $f \in B(P^{(1)})$, then $T^i(f) \in B(P^{(i)})$. The sequence $(P^{(s)})$ is called the expansion of f by the algorithm T.

Let $\beta(P) = (B_{i,j})$ be the inverse matrix of $\alpha(P)$, we set

$$\beta(P^{(1)}, \ldots, P^{(s)}) = \beta(P^{(1)}) \cdots \beta(P^{(s)}) = (B_{ij}^{(s)})_{0\leq i,j\leq n} \ ,$$

then $f^{(s)} = T^s(f)$ if, and only if,

$$f_i = \frac{B_{i0}^{(s)} + \sum_{g=1}^{n} B_{ig}^{(s)} f_g^{(s)}}{B_{00}^{(s)} + \sum_{g=1}^{n} B_{0g}^{(s)} f_g^{(s)}}, \quad \text{for all } 0 \leq i \leq n.$$

The algorithm T is said *convergent*, if for all $f \in B$,

$$\lim_{s\to+\infty}\left(\frac{B_{10}^{(s)}}{B_{00}^{(s)}},\ldots,\frac{B_{n0}^{(s)}}{B_{00}^{(s)}}\right)=f. \qquad (1)$$

The vectors $(\frac{B_{10}^{(s)}}{B_{00}^{(s)}},\ldots,\frac{B_{n0}^{(s)}}{B_{00}^{(s)}})$ are the *convergents* of f. The algorithm T is said to be *strong convergent*, if, for all $f \in B$,

$$\lim_{s\to+\infty}(B_{00}^{(s)}f-(B_{10}^{(s)},\ldots,B_{n0}^{(s)}))=0. \qquad (2)$$

4 The Jacobi-Perron Algorithm

4.1 Definition

Let $M_0 = \{\omega \in \mathbb{F}_q((X^{-1}))^n : \mid \omega \mid < 1\}$ and $M = M_0\backslash\{0\} \times M_0^{n-1}$ the homogenous version of *Jacobi − Perron algorithm* is defined, for all $f = (f_1,\ldots,f_n) \in M$, by

$$J(f)=(\frac{f_2}{f_1}-\left[\frac{f_2}{f_1}\right],\ldots,\frac{f_n}{f_1}-\left[\frac{f_n}{f_1}\right],\frac{1}{f_1}-\left[\frac{1}{f_1}\right])\in M_0^n. \qquad (3)$$

We set $M^* = \{f \in M : J^s(f)$ is well defined for all $s \geq 1\}$. Let $f \in M^*$, we define $f^{(t)} = (f_1^{(t)},\ldots,f_n^{(t)}) = J^t(f)$ and $k^{(t)}(f) = (k_1^{(t)},\ldots,k_n^{(t)}) = (\left[\frac{f_2^{(t-1)}}{f_1^{(t-1)}}\right],\ldots,\left[\frac{1}{f_1^{(t-1)}}\right])$, for all $t \geq 1$. We notice that $M^\star = \{f \in M :$ for all $s \geq 1$, $f_1^{(s)} \neq 0\}$. We claim that the sequence $(k^{(t)})$ is unique. In fact, if $f \in M^*$, we have $\frac{1}{f_1^{(t)}} = \left[\frac{1}{f_1^{(t)}}\right] + \left\{\frac{1}{f_1^{(t)}}\right\}$, let $k_n^{(t+1)} = \left[\frac{1}{f_1^{(t)}}\right]$ and $f_n^{(t+1)} = \left\{\frac{1}{f_1^{(t)}}\right\}$. As $\mathbb{F}_q(X^{-1}) = \mathbb{F}_q[X]\bigoplus M_0$, the decomposition is unique. The relation $f_i^{(t+1)} = \frac{f_{i+1}^{(t)}}{f_1^{(t)}} - \left[\frac{f_{i+1}^{(t)}}{f_1^{(t)}}\right]$ defines in an unique way $f_i^{(t+1)}$ and $k_i^{(t+1)}$, for all $1 \leq i \leq n-1$, therefore $k_n^{(t+1)}$ has an unique value, so the sequence $(k^{(t)})$ is unique.

4.2 The matrix of the homogeneous version

Let $f \in M^\star$, for all $t \geq 1$, we denote by $(\beta(k^{(t)}))$ the $(n+1)\times(n+1)$ matrix with entries in $\mathbb{F}_q[X]$

$$\beta(k^{(t)}) = \begin{pmatrix} k_n^{(t)} & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ k_1^{(t)} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k_{n-1}^{(t)} & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

We call the sequence $(k^{(t)})_{t\geq 1}$ the *expansion* of f by J, where $k^{(t)} = k^{(t)}(f)$. Of course the expansion is unique. We set

$$(B_{ij}^{(t)}) = \beta(k^{(1)}) \cdots \beta(k^{(t)})$$

whose entries satisfy the following recurrent relations, for $0 \leq i \leq n$,

$$\begin{array}{rcl} B_{i0}^{(t+1)} & = & k_n^{(t+1)} B_{i0}^{(t)} + B_{i1}^{(t)} + \ldots + k_{n-1}^{(t+1)} B_{in}^{(t)} \\ B_{i1}^{(t+1)} & = & B_{i2}^{(t)} \\ & \vdots & \\ B_{in-1}^{(t+1)} & = & B_{in}^{(t)} \\ B_{in}^{(t+1)} & = & B_{i0}^{(t)}. \end{array} \tag{4}$$

Let $f^{(s)} = J^s(f)$, then, for all $1 \leq i \leq n$,

$$f_i(B_{00}^{(s)} + \sum_{g=1}^{n} B_{0g}^{(s)} f_g^{(s)}) = B_{i0}^{(s)} + \sum_{g=1}^{n} B_{ig}^{(s)} f_g^{(s)}. \tag{5}$$

4.3 Exponential strong convergence

We can reduce the 3-parameter set $\{B_{ij}^{(t)}\}$ to a 2-parameter set $\{A_i^{(s)}\}$ by

$$A_i^{(t+j)} = B_{ij}^{(t)}, \text{ for all } 0 \leq i \leq n \text{ and } 0 \leq j \leq n,$$

$$A_i^{(t+n+1)} = B_{i0}^{(t)}, \text{ for all } 0 \leq i \leq n. \tag{6}$$

We can verify that (4) and (6) give that, for all $1 \leq i \leq n$ and $t \geq 2$,

$$A_i^{(t+n+1)} = A_i^{(t)} + \sum_{g=1}^{n} A_i^{(t+g)} k_g^{(t)}. \tag{7}$$

Now we are ready to prove one of our main results (Theorem 1.1).

Proof. Let $f = (f_1, \dots, f_n) \in M^\star$ and $f^{(t)} = J^t(f)$. By (5) and (7), we have, for all $1 \le i \le n$,

$$f_i(A_0^{(t+n+1)} + \sum_{g=1}^{n} A_0^{(t+g)} f_g^{(t)}) = A_i^{(t+n+1)} + \sum_{g=1}^{n} A_i^{(t+g)} f_g^{(t)} \qquad \text{or}$$

$$f_i A_0^{(t+n+1)} - A_i^{(t+n+1)} = \sum_{g=1}^{n} (A_i^{(t+g)} - A_0^{(t+g)} f_i) f_g^{(t)}. \tag{8}$$

Thus

$$\begin{aligned} \mid f_i A_0^{(t+n+1)} - A_i^{(t+n+1)} \mid \quad &= \mid \textstyle\sum_{g=1}^{n} (A_i^{(t+g)} - A_0^{(t+g)} f_i) f_g^{(t)} \mid \\ &\le \max_{0 \le g \le n} \mid f_g^{(t)} \mid\mid A_i^{(t+g)} - A_0^{(t+g)} f_i \mid \\ &\le e^{-1} \max_{0 \le g \le n} \mid A_i^{(t+g)} - A_0^{(t+g)} f_i \mid \end{aligned}$$

which implies that $A_0^{(t)} f_i - A_i^{(t)}$ converges exponentially to 0, as $t \to +\infty$. $\diamondsuit$

4.4 An application

Let $n \ge 1$ and $\omega \in \mathbb{F}_2((X^{-1}))$ verify the cubic equation

$$\omega^3 = X^{3n} + 1.$$

We will show that the expansion of the vector $\omega^{(0)} = (\frac{1}{\omega}, \frac{1}{\omega^2}) \in M$ is ulteriorly periodic.

We remark that $(\omega - X^n)(\omega^2 + X^{2n} + X^n\omega) = 1$ which implies that

$$\omega = X^n + \frac{1}{\omega^2 + X^{2n} + X^n\omega} \quad \text{and} \quad \omega^2 = X^{2n} + \frac{1}{\omega^4 + X^{4n} + X^{2n}\omega^2},$$

thus $[\omega] = X^n$ and $[\omega^2] = X^{2n}$. We can apply J at $\omega^{(0)}$ and denote $\omega^{(t)} = J^t(\omega^{(0)})$.

We have $\omega^{(1)} = (\frac{1}{\omega} - \left[\frac{1}{\omega}\right], \ \omega - [\omega]) = (\frac{1}{\omega}, \ \omega - X^n)$. As well as we remark that

$$\omega = X^n + \frac{1}{X^{2n}} + \dots \tag{9}$$

which implies that $[X^n\omega] = X^{2n}$ and $[\omega(\omega - X^n)] = 0$, then
$\omega^{(2)} = (\omega(\omega - X^n) - [\omega(\omega - X^n)], \ \omega - [\omega]) = (\omega(\omega - X^n), \ \omega - X^n)$.

As $\dfrac{1}{\omega(\omega - X^n)} = \dfrac{\omega^2 + X^n\omega + X^{2n}}{\omega} = \omega + X^n + \dfrac{X^{2n}}{\omega}$ and $\dfrac{1}{\omega} = \dfrac{1}{X^n} + \dfrac{1}{X^{4n}} + \dots$, so $\left[\dfrac{X^{2n}}{\omega}\right] = X^n$ and $\left[\dfrac{1}{\omega(\omega - X^n)}\right] = X^n$ which give

$$\omega^{(3)} = (\frac{1}{\omega} - \left[\frac{1}{\omega}\right],\ \frac{1}{\omega(\omega - X^n)} - \left[\frac{1}{\omega(\omega - X^n)}\right]) = (\frac{1}{\omega},\ \frac{\omega^2 + X^{2n}}{\omega}).$$

Repeated iterations of the algorithm give

$$\begin{aligned}
\omega^{(4)} &= (\omega^2 + X^{2n} - [\omega^2 + X^{2n}],\ \omega - [\omega]) = (\omega^2 + X^{2n}, \omega - X^n),\\
\omega^{(5)} &= (\frac{1}{\omega - X^n} - \left[\frac{1}{\omega - X^n}\right], \frac{1}{\omega^2 + X^{2n}} - \left[\frac{1}{\omega^2 + X^{2n}}\right])\\
&= (\omega(\omega - X^n),\ \omega^2(\omega - X^n)^2),\\
\omega^{(6)} &= (\omega(\omega - X^n) - [\omega(\omega - X^n)],\ \frac{1}{\omega(\omega - X^n)} - \left[\frac{1}{\omega(\omega - X^n)}\right])\\
&= (\omega(\omega - X^n),\ \frac{\omega^2 + X^{2n}}{\omega}) \qquad \text{and}\\
\omega^{(7)} &= (\frac{\omega - X^n}{\omega^2} - \left[\frac{\omega - X^n}{\omega^2}\right],\ \frac{1}{\omega(\omega - X^n)} - \left[\frac{1}{\omega(\omega - X^n)}\right])\\
&= (\frac{\omega - X^n}{\omega^2},\ \frac{\omega^2 + X^{2n}}{\omega}).
\end{aligned}$$

As $\dfrac{\omega^2}{\omega - X^n} = \omega^2(\omega^2 + X^n\omega + X^{2n}) = (X^{3n} + 1)\omega + X^n\omega^3 + X^{2n}\omega^2$, and, by (9), $[X^{3n}\omega] = X^{4n} + X^n$, so $\left[\dfrac{\omega^2}{\omega - X^n}\right] = X^{4n} + X^n = X^n\omega^3$. Thus

$$\omega^{(8)} = (\omega(\omega - X^n) - [\omega(\omega - X^n)],\ \frac{\omega^2}{\omega - X^n} - \left[\frac{\omega^2}{\omega - X^n}\right]) = (\omega(\omega - X^n),\ \omega^2(\omega^2 - X^{2n}))$$

which gives $\omega^{(5)} = \omega^{(8)}$. Hence, the expansion is ulteriorly periodic.

5 The version of J introduced by Dubois

In order to have an algorithm defined in the whole space E (see *Section* 2) a version of (3) is introduced in [D1]. For all $f = (f_1, \ldots, f_n) \in E$, $f_1 \notin \mathbb{F}_q[X]$, we define the map

$$D(f) = \left(\frac{f_2 - [f_2]}{f_1 - [f_1]}, \ldots, \frac{f_n - [f_n]}{f_1 - [f_1]}, \frac{1}{f_1 - [f_1]}\right) \in E. \tag{10}$$

Let $f \in E^\star = \{f \in E\ :\ D^s(f)$ is well defined for all $s \geq 1\}$. For all $t \geq 1$, we set $f^{(t)} = D^t(f) = (f_1^{(t)}, \ldots, f_n^{(t)})$ and

$$a^{(t)}(f) = (a_1^{(t)}, \ldots, a_n^{(t)}) = ([f_2^{(t-1)}], \ldots, [f_n^{(t-1)}], [f_1^{(t-1)}]). \tag{11}$$

The latter sequence is the *expansion* of f by D. Using D we obtain

$$\begin{cases} f_1^{(t)} = a_1^{(t)} + \dfrac{1}{f_n^{(t+1)}} \text{ and } \mid f_n^{(t+1)} \mid > 1, \\ \\ f_i^{(t)} = a_i^{(t)} + \dfrac{f_{i-1}^{(t+1)}}{f_n^{(t+1)}} \text{ and } \mid f_n^{(t+1)} \mid > \mid f_{i-1}^{(t+1)} \mid, \text{ for all } 2 \leq i \leq n. \end{cases} \quad (12)$$

It is clear from (11) that $\mid a_n^{(t)} \mid > 1$ and $\mid a_n^{(t)} \mid > \mid a_i^{(t)} \mid$, for all $1 \leq i \leq n-1$. We define the collection of polynomials $\{H_i^{(t)}\}$ by

$$\begin{cases} H_i^{(j)} = 0 \text{ , if } i \neq j \text{ and } H_i^{(i)} = 1, \text{ for all } 0 \leq i, j \leq n, \\ \\ H_i^{(t+n+1)} = H_i^{(t)} + a_1^{(t)} H_i^{(t+1)} + \ldots + a_n^{(t)} H_i^{(t+n)}, \text{ for all } 0 \leq i \leq n \text{ and } t \geq 0. \end{cases}$$

The strong convergence of D is showed in [D1], for completeness a proof is included here.

Theorem 5.1 (Dubois) *Let $f \in E^\star$, then, for all $1 \leq i \leq n$,*

$$f_i H_0^{(t)} - H_i^{(t)} \quad \textit{converges exponentially to } 0, \quad \textit{as } t \to +\infty.$$

Proof. Let $f = (f_1, ..., f_n) \in E^\star$ and $f^{(t)} = D^t(f)$, then, for all $1 \leq i \leq n$,

$$f_i(H_0^{(t)} + \sum_{g=1}^{n} H_0^{(t+g)} f_g^{(t)}) = H_i^{(t)} + \sum_{g=1}^{s} n H_i^{(t+g)} f_g^{(t)}$$

which implies that

$$H_i^{(t+n)} - f_i H_0^{(t+n)} = (f_i H_0^{(t)} - H_i^{(t)}) \frac{1}{f_n^{(t)}} + \sum_{g=1}^{n-1} (f_i H_0^{(t+g)} - H_i^{(t+g)}) \frac{f_g^{(t)}}{f_n^{(t)}}.$$

As $\mid f_n^{(t)} \mid > 1$ and $\mid f_n^{(t)} \mid > \mid f_i^{(t)} \mid$ (5.3), for all $1 \leq i \leq n-1$,

$$\begin{aligned} \mid f_i H_0^{(t+n)} - H_i^{(t+n)} \mid \quad &= \mid (f_i H_0^{(t)} - H_i^{(t)}) \frac{1}{f_n^{(t)}} + \sum_{g=1}^{n-1} (H_i^{(t+g)} - H_0^{(t+g)} f_i) f_g^{(t)} \mid \\ &\leq \mid \frac{1}{f_n^{(t)}} \mid\mid f_i H_0^{(t)} - H_i^{(t)} \mid + \max_{1 \leq g \leq n-1} \mid \frac{f_g^{(t)}}{f_n^{(t)}} \mid\mid f_i H_0^{(t+g)} - H_i^{(t+g)} \mid \\ &\leq e^{-1} \max_{0 \leq g \leq n-1} \mid H_0^{(t+g)} f_i - H_i^{(t+g)} \mid, \end{aligned}$$

thus $\lim\limits_{t\to+\infty}(f_iH_0^{(t)} - H_i^{(t)}) = 0$, for all $1 \le i \le n$, and the convergence is exponential. ♢

Next we establish a relation between the algorithms J and D.

Lemma 5.2 *i) Let $P \in \mathbb{F}_q[X]^n$ and $f \in E$, $f_1 \notin \mathbb{F}_q[X]$, then $D(P+f) = D(f)$.*
ii) For all $f \in M^\star$ and $k \ge 1$,

$$D \circ J^k(f) = D \circ D^k(f) \text{ and } J^k(f) = \{D^k(f)\}.$$

Proof. i) Let $f = (f_1, \dots, f_n)$ and $P = (P_1, \dots, P_n) \in \mathbb{F}_q[X]^n$, we have

$$D(P+f) = (\frac{f_2 - [f_2]}{f_1 - [f_1]}, \dots, \frac{f_n - [f_n]}{f_1 - [f_1]}, \frac{1}{f_1 - [f_1]}) = D(f).$$

ii) Let $f = (f_1, \dots, f_n) \in M^\star$, we have

$$D(f) = (\frac{f_2}{f_1}, \dots, \frac{f_n}{f_1}, \frac{1}{f_1}) \text{ and } J(f) = (\frac{f_2}{f_1} - \left[\frac{f_2}{f_1}\right], \dots, \frac{f_n}{f_1} - \left[\frac{f_n}{f_1}\right], \frac{1}{f_1} - \left[\frac{1}{f_1}\right]),$$

thus $D \circ J(f) = D \circ D(f)$. By induction we obtain $D \circ J^k(f) = D \circ D^k(f)$, for all $k \ge 1$.

On the other hand, we have

$$J(f) = (\{\frac{f_2}{f_1}\}, \dots, \{\frac{f_n}{f_1}\}, \{\frac{1}{f_1}\}) = \{D(f)\}.$$

By induction, we obtain $J^k(f) = \{D^k(f)\}$, for all $k \ge 1$. ♢

Next we relate the periodic properties of J (3) and D (10).

Theorem 5.3 *Let $f \in M^\star$.*
i) If the expansion of f by D is periodic, then its expansion by J is also periodic.
ii) If the expansion of f by J is periodic, then the expansion of $D(f)$ by D is periodic.

Proof. i) If the expansion of $f \in M^*$ by D is periodic, then there exists $p \ge 1$ such that $D^{p+k}(f) = D^k(f)$, for all $k \ge 1$. By Lemma 5.2 *ii*), we conclude that

$$J^{p+k}(f) = \{D^{p+k}(f)\} = \{D^k(f)\} = J^k(f),$$

hence, the expansion of f by J is periodic.

ii) If the expansion of f is periodic by J, then there exists $p \geq 1$ such that $J^{p+k}(f) = J^k(f)$, for all $k \geq 0$. As we have,

$$D \circ J^{p+k}(f) = D \circ D^{p+k}(f) = D^{p+k+1}(f) \text{ and } D \circ J^{p+k}(f) = D \circ J^k(f) = D \circ D^k(f),$$

so from Lemma 5.2 ii), we obtain $D^{k+1}(f) = D^{p+k+1}(f)$ which gives that the expansion of $D(f)$ by D is periodic. ♢

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