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On Rational Solutions of the Diophantine Equations $X^2 - (t^2 - t)Y^2 - (4t - 2)X + (4t^2 - 4t)Y = 8$

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Abstract

Let $t \geq 2$ be a positive integer. This paper deals with some formulas for the integer solutions of the Diophantine equation $(E)\ :\ X^2 - (t^2 - t)Y^2 - (4t - 2)X + (4t^2 - 4t)Y = 8$.

Keywords: Pell's equation, solutions of Diophantine equations.

1 Introduction

Diophantus in his Aritmetika deals with the search for rational (not necessarily integral) solutions of special types of Diophantine equations. The general theory of solving of Diophantine equations of the first degree was developed by C.G. Bachet in the 17th century; for more details on this subject see Linear equation. P. Fermat, J. Wallis, L. Euler, J.L. Lagrange, and C.F. Gauss in the early 19th century mainly studied Diophantine equations of the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

where $a,\ b,\ c,\ d$ and f are integers, i.e. general inhomogeneous equations of the second degree with two unknowns. Lagrange used continued fractions in his study of general inhomogeneous Diophantine equations of the second degree with two unknowns. Gauss developed the general theory of quadratic forms, which is the basis of solving certain types of Diophantine equations.

In this paper, we will proceed to study rational solutions of the Diophantine equation

$$(E)\ :\ X^2-(t^2-t)Y^2-(4t-2)X+(4t^2-4t)Y=8$$

where, $t\geq 1$ be an integer. Here and in what follows we shall mean by a solution a rational solution.

2 Main results

In this section, we consider the solutions of the Diophantine equation

$$(E)\ :\ X^2-(t^2-t)Y^2-(4t-2)X+(4t^2-4t)Y=8$$

where, $t\geq 1$ and $l\geq 1$ be two integers.

Theorem 2.1 $(X_1,Y_1)=(8t-4,8)$ *is the fundamental solution of the Diophantine equation* (E).

Proof. It is easily seen that $(X_1,Y_1)=(8t-4,8)$ is a solution of the Diophantine equation (E), since $(8t-4)^2-(t^2-t)(8)^2-(4t-2)(8t-4)+(4t^2-4t)8=8$.

Theorem 2.2 *Let*

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}=\begin{pmatrix} 2t-1 & 2(t^2-t) \\ 2 & 2t-1 \end{pmatrix}^n\begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{1}$$

for $n\geq 1$. *Then* $(X_n,Y_n)=\left(\frac{u_n}{3^{n-1}}+2t-1,\frac{v_n}{3^{n-1}}+2\right)$ *is a solution of* (E).

To prove this theorem, we need the following results.

Theorem 2.3 *Let* (x_1,y_1) *be the fundamental solution of the Pell equation* $x^2-Dy^2=9$, *and let*

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}=\begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^n\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2}$$

for $n\geq 1$. *Then the integer solutions of the Pell equation* $x^2-Dy^2=9$ *are* (x_n,y_n), *where*

$$(x_n,y_n)=\left(\frac{u_n}{3^{n-1}},\frac{v_n}{3^{n-1}}\right) \tag{3}$$

Proof. We prove the theorem using the method of mathematical induction. For $n = 1$, we have from (2), $(u_1, v_1) = (x_1, y_1)$ which is the fundamental solution of $x^2 - Dy^2 = l^2$. Now, we assume that the Pell equation $x^2 - Dy^2 = 9$ is satisfied for (x_{n-1}, y_{n-1}), i.e.

$$x_{n-1}^2 - Dy_{n-1}^2 = \frac{u_{n-1}^2 - Dv_{n-1}^2}{k^{2n-4}} = 9 \tag{4}$$

and we show that it holds for (x_n, y_n).

Indeed, by (2), it is easy to prove that

$$\begin{cases} u_n = x_1 u_{n-1} + D y_1 v_{n-1} \\ v_n = y_1 u_{n-1} + x_1 v_{n-1} \end{cases} \tag{5}$$

Hence,

$$\begin{aligned} x_n^2 - Dy_n^2 &= \frac{u_n^2 - Dv_n^2}{3^{2n-2}} \\ &= \frac{(x_1 u_{n-1} + D y_1 v_{n-1})^2 - D(y_1 u_{n-1} + x_1 v_{n-1})^2}{3^{2n-2}} \\ &= \frac{x_1^2 u_{n-1}^2 + 2x_1 u_{n-1} D y_1 v_{n-1} + D^2 y_1^2 v_{n-1}^2}{3^{2n-2}} \\ &\quad - \frac{D(y_1^2 u_{n-1}^2 + 2y_1 u_{n-1} x_1 v_{n-1} + x_1^2 v_{n-1}^2)}{3^{2n-2}} \\ &= \frac{x_1^2(u_{n-1}^2 - Dv_{n-1}^2) - Dy_1^2(u_{n-1}^2 - Dv_{n-1}^2)}{3^{2n-2}} \\ &= (x_1^2 - Dy_1^2)\frac{(u_{n-1}^2 - Dv_{n-1}^2)}{3^{2n-2}} \end{aligned}$$

Applying (4), it is easily seen that

$$u_{n-1}^2 - Dv_{n-1}^2 = k^{2n-4} k^2 = 3^{2n-2}.$$

Hence we conclude that

$$x_n^2 - Dy_n^2 = (x_1^2 - Dy_1^2) = 9.$$

Therefore (x_n, y_n) is also a solution of the Pell equation $x^2 - Dy^2 = 9$. Since n is arbitrary, we get all integer solutions of the Pell equation $x^2 - Dy^2 = 9$.

Lemma 2.4 $(x_1, y_1) = (6t - 3, 6)$ *is the fundamental solution of the Pell equation* $x^2 - Dy^2 = 9$.

Proof. (of **Theorem 2.2)** we have

$$\begin{aligned} & X^2-(t^2-t)Y^2-(4t-2)X+(4t^2-4t)Y \\ = & \left(\frac{u_n}{3^{n-1}}+2t-1\right)^2-(t^2-t)\left(\frac{v_n}{3^{n-1}}+2\right)^2-(4t-2)\left(\frac{u_n}{3^{n-1}}+2t-1\right) \\ + & (4t^2-4t)\left(\frac{v_n}{3^{n-1}}+2\right) \\ = & \left(\frac{u_n}{3^{n-1}}\right)^2-(t^2-t)\left(\frac{v_n}{3^{n-1}}\right)^2-1=8. \end{aligned}$$

Theorem 2.5 *The solutions* (X_n, Y_n) *of the Diophantine equation* (E), *satisfy the recurrence relations*

$$\begin{cases} X_n = \dfrac{(2t-1)X_{n-1}+2(t^2-t)Y_{n-1}-8t^2+8t-1}{3}+2t-1 \\ \\ Y_n = \dfrac{2X_{n-1}+(2t-1)Y_{n-1}-8t+4}{3}+2 \end{cases} \qquad (6)$$

Proof. It is clear that $u_n=(2t-1)u_{n-1}+2(t^2-t)v_{n-1}$. Then, we get

$$3^{n-1}(x_n-(2t-1))=(2t-1)3^{n-2}(x_{n-1}-(2t-1))+23^{n-2}(t^2-t)(y_{n-1}-2),$$

which gives

$$3(x_n-(2t-1))=(2t-1)(x_{n-1}-(2t-1))+2(t^2-t)(y_{n-1}-2),$$

hence

$$X_n=\frac{(2t-1)X_{n-1}+2(t^2-t)Y_{n-1}-8t^2+8t-1}{3}+2t-1.$$

And, similarly, we get

$$Y_n=\frac{2X_{n-1}+(2t-1)Y_{n-1}-8t+4}{3}+2$$

2.1 Algorithm

function $[X, Y] =$ solution (n, t)

$X = [(8t - 4)];$

$Y = [8];$

$for \quad i = 2 : n$

$$A = \frac{(2t-1)X_{i-1} + 2(t^2 - t)Y_{i-1} - 8t^2 + 8t - 1}{3} + 2t - 1;$$

$$B = \frac{2X_{i-1} + (2t-1)Y_{i-1} - 8t + 4}{3} + 2;$$

$X = [X, A]; Y = [Y, B];$

end

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