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On quadratic Diophantine Equations $x^2 - (t^2 - t)y^2 - ((2p)^2 t - 2p)x + ((2p)^2 t^2 - (2p)^2 t)y = 0$

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Abstract

Let $t \geq 2$ be a positive integer and $p \geq 1$ be any fixed integer. Extending the work of A. Tekcan, here we consider the number of integer solutions of Diophantine equation $E\ :\ x^2 - (t^2 - t)y^2 - ((2p)^2 t - 2p)x + ((2p)^2 t^2 - (2p)^2 t)y = 0$. We also obtain some formulas and recurrence relations on the integer solution (x_n, y_n) of E. Equally, we are able to solve the equation $\widetilde{E}\ :\ u^2 - (t^2 - t)v^2 = p^2 + 4t(p^4 - p^3)$ which is a simple transformation of E via an appropriate transformation T.

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1 Introduction

A Diophantine equation is an algebraic equation whose coefficients lie in the ring $\mathbb{Z}$ of rational integers and whose solutions are sought in that ring. The name comes from Diophantus, an Alexandrian mathematician of the third century A.D., who proposed many Diophantine problems ; but such equations have a very long history, extending back to ancient Egypt, Babylonia, and Greece. In general, the Diophantine equation is an equation given by

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

The equation

$$x^2 - dy^2 = 1$$

is a special case of Diophantine equation, known as the Pell equation.

Pell's equation $x^2 - Dy^2 = 1$ was solved by Lagrange in terms of simple continued fractions. Lagrange was the first to prove that $x^2 - Dy^2 = 1$ has infinitly many solutions in integers if $D \neq 1$ is a fixed positive non-square integer. If the lenght of the periode of $\sqrt{D}$ is l, all positive solutions are given by $x = P_{2vk-1}$ and $y = Q_{2vk-1}$ if k is odd, and by $x = P_{vk-1}$ and $y = Q_{vk-1}$ if k is even, where $v = 1, 2, \ldots$ and $\frac{P_n}{Q_n}$ denotes the nth convergent of the continued fraction expansion of $\sqrt{D}$. Incidentally, $x = P_{(2v-1)(k-1)}$ and $y = Q_{(2v-1)(k-1)}$, $v = 1, 2, \ldots$, are the positive solutions of $x^2 - Dy^2 = -1$ provided that l is odd.

There is no solution of $x^2 - Dy^2 = \pm 1$ other than $x_v, y_v \ : \ v = 1, 2, \ldots$ given by $(x_1 + \sqrt{D}y_1)^v = x_v + \sqrt{D}y_v$, where x_1, y_1 is the least positive solution called the fundamental solution, which there are different method for finding it. The reader can find many references in the subject in the book [9].

For completeness we recall that there are many papers in which are considered different types of Pell's equation. Many authors such as Tekcan [3], Kaplan and Williams [4], Matthews [5], Mollin, Poorten and Williams [6], Stevenhagen [7] and the others consider eome specific Pell equations and their integer solutions. A. Tekcan in [3], considered the equation $x^2 - Dy^2 = \pm 4$, and he obtained some formulas for its integer solutions. He mentioned two conjecture which was proved by A. S. Shabani [8]. In [1], we considered the Pell equation $x^2 - Dy^2 = \pm k^2$ when $D \neq 1$ be a positive non-square and $k \geq 2$, we obtain some formulas for its integer solutions.

2 The Diophantine Equation $x^2 - (t^2 - t)y^2 - ((2p)^2 t - 2p)x + ((2p)^2 t^2 - (2p)^2 t)y = 0$

Let $t \geq 2$ be an integer. In [2], A. Tekcan consider the number of integer solutions of Diophantine equation $D \ : \ x^2 - (t^2 - t)y^2 - (4t - 2)x + (4t^2 - 4t)y = 0$ over $\mathbb{Z}$. He also derive some recurrence relations on the integer solutions (x_n, y_n) of D. In the present paper, we consider the integer of Diophantine equation

$$E \ : \ x^2 - (t^2 - t)y^2 - ((2p)^2 t - 2p)x + ((2p)^2 t^2 - (2p)^2 t)y = 0 \qquad (1)$$

over $\mathbb{Z}$, where $t \geq 2$ and $p \geq 1$ be two integers.

Note that the resolution of E in its present form is very difficult, that is, we can not determine how many solutions E has and what they are. So, we have to transform E into an appropriate Diophantine equation which can be easily solved. To get this let

$$T \ : \ \begin{cases} x = u + h \\ y = v + k \end{cases} \qquad (2)$$

be a translation for some h and k.

By applying the transformation T to E, we get

$$T(E) := \widetilde{E}: \quad \begin{array}{l}(u+h)^2 - (t^2-t)(v+k)^2 - ((2p)^2t - 2p)(u+h) \\ +((2p)^2t^2 - (2p)^2t)(v+k) = 0\end{array} \tag{3}$$

In (3), we obtain $u(2h+2p-(2p)^2t)$ and $v(-2kt^2+2kt+(2p)^2t^2-(2p)^2t)$. So we get $h = 2p^2t - p$ and $k = 2p^2$. Consequently for $x = u + 2p^2t - p$ and $y = v + 2p^2$, we have the Diophantine equation

$$\widetilde{E} \ : \ u^2 - (t^2-t)v^2 = p^2 + 4t(p^4 - p^3) \tag{4}$$

which is a Pell equation.

It is clear that D, considered in [2], is a particular case of E ($p = 1$).

Now, we try to find all integer solutions (u_n, v_n) of $T(E)$ and then we can retransfer all results from $T(E)$ to E by using the inverse of T.

Theorem 2.1 *Let $\widetilde{E}$ be the Diophantine equation in (3), then*

(1) The fundamental solution of $\widetilde{E}$ is $(u_1, v_1) = (2p^2t - p, 2p^2)$.

(2) Define the sequence (u_n, v_n) by

$$\begin{cases} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 2p^2t - p \\ 2p^2 \end{pmatrix} \\ \\ \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} 2t-1 & 2t^2-2t \\ 2 & 2t-1 \end{pmatrix}^{n-1} \begin{pmatrix} u_1 \\ v_2 \end{pmatrix}, \ \forall n \geq 2. \end{cases} \tag{5}$$

Then (u_n, v_n) is a solution of $\widetilde{E}$.

(3) The solutions (u_n, v_n) satisfy the recurrence relations

$$\begin{cases} u_n = (2t-1)u_{n-1} + (2t^2-2t)v_{n-1} \\ \\ v_n = 2u_{n-1} + (2t-1)v_{n-1} \end{cases} \tag{6}$$

for $n \geq 2$

(4) The solutions (u_n, v_n)

$$\begin{cases} u_n = (4t-3)(u_{n-1} + u_{n-2}) - u_{n-3} \\ \\ v_n = (4t-3)(v_{n-1} + v_{n-2}) - v_{n-3} \end{cases} \tag{7}$$

for $n \geq 4$
(5) The n-th solution (u_n, v_n) *can be given by*

$$\frac{u_n}{v_n} = \left[t-1; \underbrace{2, 2t-2, \cdots, 2, 2t-2}_{n-1\ times}, 1, 2p-1 \right], \forall n \geq 1. \tag{8}$$

Proof.
(1) It is easily seen that $(u_1, v_1) = (2p^2t - p, 2p^2)$ is the fundamental solution of $\widetilde{E}$, since $(2p^2t-p)^2 - (t^2-t)(2p^2)^2 = p^2 + 4t(p^4 - p^3)$.

(2) We prove it using the method of mathematical induction. Let $n = 1$, by (5) we get $(u_1, v_1) = (2p^2t - p, 2p^2)$ which is the fundamental solution and so is a solution of $\widetilde{E}$. Now, we assume that the Diophantine equation (4) is satisfied for n, that is $\widetilde{E}\ :\ u_n^2 - (t^2-t)v_n^2 = p^2 + 4t(p^4 - p^3)$. We try to show that this equation is also satisfied for $n+1$. Applying (5), we find that

$$\begin{aligned}
\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} &= \begin{pmatrix} 2t-1 & 2t^2-2t \\ 2 & 2t-1 \end{pmatrix}^n \begin{pmatrix} u_1 \\ v_2 \end{pmatrix} \\
&= \begin{pmatrix} 2t-1 & 2t^2-2t \\ 2 & 2t-1 \end{pmatrix} \begin{pmatrix} 2t-1 & 2t^2-2t \\ 2 & 2t-1 \end{pmatrix}^{n-1} \begin{pmatrix} u_1 \\ v_2 \end{pmatrix} \\
&= \begin{pmatrix} 2t-1 & 2t^2-2t \\ 2 & 2t-1 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} \\
&= \begin{pmatrix} (2t-1)u_n + (2t^2-2t)v_n \\ 2u_n + (2t-1)v_n \end{pmatrix}
\end{aligned} \tag{9}$$

Hence, we conclude that

$$\begin{aligned}
u_{n+1}^2 - (t^2-t)v_{n+1}^2 &= \left[(2t-1)u_n + (2t^2-2t)v_n\right]^2 - (t^2-t)\left[2u_n + (2t-1)v_n\right]^2 \\
&= u_n^2 - (t^2-t)v_n^2 = p^2 + 4t(p^4 - p^3).
\end{aligned}$$

So (u_{n+1}, v_{n+1}) is also solution of $\widetilde{E}$.

(3) Using (9), we find that

$$\begin{cases} u_n = (2t-1)u_{n-1} + (2t^2-2t)v_{n-1} \\ \\ v_n = 2u_{n-1} + (2t-1)v_{n-1} \end{cases}$$

for $n \geq 2$

(4) We prove it using the method of mathematical induction. For $n = 4$, we get
$u_1 = 2p^2t - p$
$u_2 = 8p^2t^2 - 2(3p^2 + p)t + p$
$u_3 = 32p^2t^2 - 8(5p^2 + p)t^2 + 2(5p^2 + 4p^2)t - p$
and
$u_4 = 128p^2t^4 - 32(7p^2 + p)t^3 + 16(7p^2 + 3p)t^2 - 2(7p^2 + 9p)t + p$. Hence

$$\begin{aligned} u_4 &= 128p^2t^4 - 32(7p^2 + p)t^3 + 16(7p^2 + 3p)t^2 - 2(7p^2 + 9p)t + p. \\ &= (4t - 3)[32p^2t3 - (32p^2 + 8p)t^2 + (4p^2 + 6p)t] - (2p^2t - p) \\ &= (4t - 3)[32p^2t^2 - 8(5p^2 + p)t^2 + 2(5p^2 + 4p^2)t - p + 8p^2t^2 - 2(3p^2 + p)t \\ &+p] - (2p^2t - p) \\ &= (4t - 3)(u_3 + u_2) - u_1. \end{aligned}$$

So $u_n = (4t-3)(u_{n-1} + u_{n-2}) - u_{n-3}$. is satisfied for $n = 4$. Let us assume that this relation is satisfied for n, that is,

$$u_n = (4t - 3)(u_{n-1} + u_{n-2}) - u_{n-3}. \tag{10}$$

Then using (9) and (10), we conclude that

$$u_{n+1} = (4t - 3)(u_n + u_{n-1}) - u_{n-2},$$

completing the proof.

Similarly, we prove that $v_n = (4t - 3)(v_{n-1} + v_{n-2}) - v_{n-3},\ \forall n \geq 4$.

(5) We prove it using the method of mathematical induction. For $n = 1$, we have

$$\frac{u_1}{v_1} = \frac{2p^2t - p}{2p^2} = t - 1 + \cfrac{1}{1 + \cfrac{1}{2p - 1}}$$

$$= [t - 1; 1, 2p - 1]$$

which is the fundamental solution of $\widetilde{E}$. Let us assume that the n-th solution (u_n, v_n) is given by

$$\frac{u_n}{v_n} = \left[t - 1; \underbrace{2, 2t - 2, \cdots, 2, 2t - 2}_{n-1\ times}, 1, 2p - 1 \right].$$

and we show that it holds for (u_{n+1}, y_{n+1}).

Using (6) , we have

$$\begin{aligned}\frac{u_{n+1}}{v_{n+1}} &= \frac{(2t-1)u_n+(2t^2-2t)v_n}{2u_n+(2t-1)v_n} \\ &= \frac{2(t-1)u_n+u_n+(2t-1)(t-1)v_n+(t-1)v_n}{2u_n+(2t-1)v_n} \\ &= t-1+\cfrac{1}{2+\cfrac{1}{t-1+\cfrac{u_n}{v_n}}}\end{aligned}$$

as

$$\begin{aligned}t-1+\frac{u_n}{v_n} &= t-1+t-1+\cfrac{1}{2+\cfrac{1}{\cdots\cdots+\cfrac{1}{2t-2+\cfrac{1}{1+\cfrac{1}{2p-1}}}}} \\ &= 2t-2+\cfrac{1}{2+\cfrac{1}{2t-2+\cfrac{1}{\cdots+\cfrac{1}{2t-2+\cfrac{1}{1+\cfrac{1}{2p-1}}}}}}\end{aligned}$$

we get

$$\begin{aligned}\frac{u_{n+1}}{v_{n+1}} &= t-1+\cfrac{1}{2+\cfrac{1}{2t-2+\cfrac{1}{2+\cfrac{1}{2t-2+\cfrac{1}{\cdots+\cfrac{1}{2t-2+\cfrac{1}{1+\cfrac{1}{2p-1}}}}}}}} \\ &= \left[t-1;\underbrace{2,2t-2,\cdots,2,2t-2}_{n\ times},1,2p-1\right].\end{aligned}$$

completing the proof.

As we reported above, the Diophantine equation E could be transformed into the Diophantine equation $\widetilde{E}$ via the transformation T. Also, we showed that $x = u + 2p^2t - p$ and $y = v + 2p^2$. So, we can retransfer all results from $\widetilde{E}$ to E by applying the inverse of T. Thus, we can give the following main theorem

Theorem 2.2 *Let D be the Diophantine equation in (1). Then*
(1) The fundamental (minimal) solution of E is $(x_1, y_1) = (4p^2t - 2p, 4p^2)$
(2) Define the sequence $\{(x_n, y_n)\}_{n\geq 1} = \{(u_n + 2p^2t - p, v_n + 2p^2)\}$, where $\{(x_n, y_n)\}$ defined in (5). Then (x_n, y_n) is a solution of E. So it has infinitely many integer solutions $(x_n, y_n) \in \mathbb{Z} \times \mathbb{Z}$.
(3) The solutions (x_n, y_n) satisfy the recurrence relations

$$\begin{cases} x_n = (2t-1)x_{n-1} + (2t^2 - 2t)y_{n-1} - 8p^2t^2 + (8p^2 + 2p)t - 2p \\ \\ y_n = 2x_{n-1} + (2t-1)y_{n-1} - 8p^2t + 4p^2 + 2p \end{cases} \tag{11}$$

for $n \geq 2$

(4) The solutions (u_n, v_n) satisfy the recurrence relations

$$\begin{cases} x_n = (4t-3)(x_{n-1} + x_{n-2}) - x_{n-3} - 16p^2t^2 + (16p^2 + 8p)t - 8p \\ \\ y_n = (4t-3)(y_{n-1} + y_{n-2}) - y_{n-3} - 16p^2t + 16p^2. \end{cases} \tag{12}$$

for $n \geq 4$

Example 2.3 *Let $t = 4$ and $p = 2$. Then $(u_1, v_1) = (30, 8)$ is the fundamental solution of*

$$\widetilde{E} \; : \; u^2 - 12v^2 = 132$$

and some other solutions are

$$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 7 & 24 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} 30 \\ 8 \end{pmatrix} = \begin{pmatrix} 402 \\ 116 \end{pmatrix}$$

$$\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 7 & 24 \\ 2 & 7 \end{pmatrix}^2 \begin{pmatrix} 30 \\ 8 \end{pmatrix} = \begin{pmatrix} 5598 \\ 1616 \end{pmatrix}$$

$$\begin{pmatrix} u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 7 & 24 \\ 2 & 7 \end{pmatrix}^3 \begin{pmatrix} 30 \\ 8 \end{pmatrix} = \begin{pmatrix} 77970 \\ 22508 \end{pmatrix}$$

$$\begin{pmatrix} u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 7 & 24 \\ 2 & 7 \end{pmatrix}^4 \begin{pmatrix} 30 \\ 8 \end{pmatrix} = \begin{pmatrix} 1085982 \\ 313496 \end{pmatrix}$$

Further

$$\frac{u_1}{v_1} = [3; , 1, 3] = \frac{30}{8}$$

$$\frac{u_2}{v_2} = [3; 2, 6, 1, 3] = \frac{402}{116}$$

$$\frac{u_3}{v_3} = [3; 2, 6, 2, 6, 1, 3] = \frac{5598}{1616}$$

$$\frac{u_4}{v_4} = [3; 2, 6, 2, 6, 2, 6, 1, 3] = \frac{77970}{22508}$$

$$\frac{u_5}{v_5} = [3; 2, 6, 2, 6, 2, 6, 2, 6, 1, 3] = \frac{1085982}{313496}$$

It can be concluded now, that the fundamental solution of

$$E \ : \ x^2 - 12y^2 - 60x + 192y = 0$$

is $(60, 16)$. *Some other solutions are*

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 432 \\ 124 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 5628 \\ 1624 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} 78000 \\ 22516 \end{pmatrix}$$

$$\begin{pmatrix} x_5 \\ y_5 \end{pmatrix} = \begin{pmatrix} 1086012 \\ 313504 \end{pmatrix}$$

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